



Techniques of Water-Resources Investigations
of the United States Geological Survey

Chapter B2

**INTRODUCTION TO
GROUND-WATER HYDRAULICS**

A Programed Text for Self-Instruction

By **Gordon D. Bennett**

Book 3

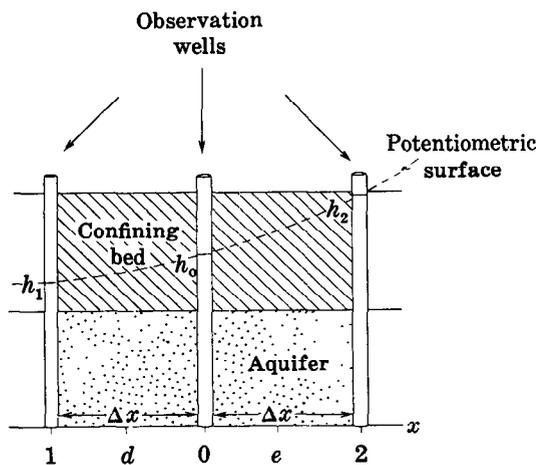
APPLICATIONS OF HYDRAULICS

Part VII. Finite-Difference Methods

Introduction

In preceding chapters we have considered formal mathematical solutions to the differential equations of ground-water flow. In practice, however, we find that such formal solutions are available only for a small minority of field problems, representing relatively simple boundary conditions. In most

cases, we are forced to seek approximate solutions, using methods other than direct formal solution. In Part VII, we consider one such method—the simulation of the differential equations by finite difference equations, which in turn can be solved algebraically or numerically.



Three observation wells tap a confined aquifer. The wells are arranged in a straight line in the x direction at a uniform spacing, Δx . The water levels in the three wells are designated h_1 , h_0 , and h_2 as indicated in the figure.

QUESTION

Which of the following equations gives a reasonable approximation for the derivative, $\partial h / \partial x$, at point d , midway between well 1 and well 0?

- | | |
|--|----|
| $\left(\frac{\partial h}{\partial x}\right)_d \approx \frac{h_1 - h_2}{\Delta x}$ | 7 |
| $\left(\frac{\partial h}{\partial x}\right)_d \approx \frac{h_2 - h_1}{2\Delta x}$ | 26 |
| $\left(\frac{\partial h}{\partial x}\right)_d \approx \frac{h_0 - h_1}{\Delta x}$ | 12 |

Turn to Section:

2 □ □

Your answer, $h_{i,j}$, in Section 3 is correct.

QUESTION

Following the same conventions, which of the following expressions would serve as a finite-difference approximation to the term

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

at the point $h_{i,j}$?

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_{i-2,j} + h_{i-1,j} + h_{i+1,j} + h_{i+2,j} - 4h_{i,j}}{a^2}$$

Turn to Section:

20

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_{i,j+1} + h_{i+1,j} + h_{i,j+2} + h_{i+2,j} - 4h_{i,j}}{a^2}$$

18

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j}}{a^2}$$

4

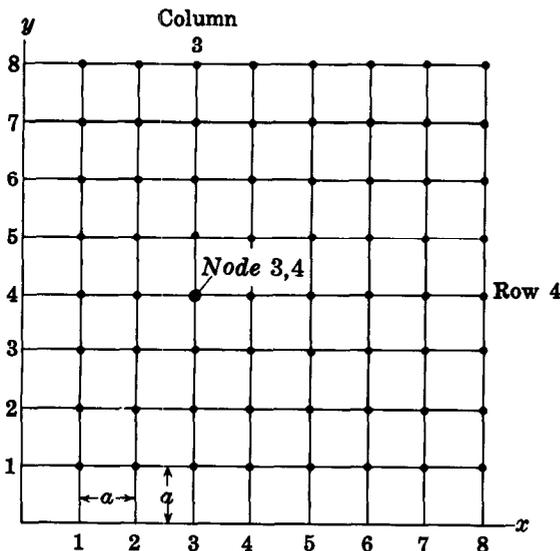
3 □ □

Your answer in Section 15,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_1 + h_2 + h_3 + h_4 - 4h_0}{a^2},$$

is correct. These approximations to $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$ can be obtained more formally through the use of Taylor series expansions. A certain error is involved in approximating the derivatives by finite differences, and we can see intuitively that this error will generally decrease as a is given smaller and smaller values.

Now let us place a rectangular grid of intersecting lines, as shown in the diagram



over the x, y plane. The lines are drawn at a uniform spacing, a , and are numbered successively from the origin. Lines parallel to the x -axis are termed rows, while lines parallel to the y -axis are termed columns. The intersections of the grid lines are termed nodes and are identified by the numbers associated with the intersecting lines. For example, the node 3, 4 is that formed by the intersection of the third column to the right of the y -axis with the fourth row above the x -axis. The spacing a , may be thought of as a unit of measurement; the node numbers then give the number of units of distance of a given node from the x and y axes. The head at a given node is indicated by using the node numbers for a subscript notation; for example, the head at node 3, 4 would be indicated by $h_{3,4}$.

QUESTION

Following this convention, how would we indicate the head at a node located i units to the right of the y axis and j units above the x axis (that is, at the point $x = i \cdot a, y = j \cdot a$, in the conventional Cartesian notation)?

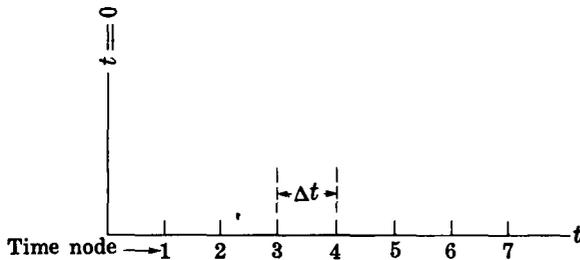
Turn to Section:

- $h_{j,i}$
- $h_{i,j}$
- $h_{ia,ja}$

- 14
- 2
- 5

4 □ □

Your answer in Section 2 is correct. We next consider the time axis and divide it as shown in the sketch into segments of length Δt , again numbering the division marks successively from $t=0$. We also introduce a third subscript, indicating the time at which a given head value is observed; for example,



$h_{i,j,n}$ refers to the head at the node i, j of the x, y plane at the time indicated by the n th division mark on the time axis.

QUESTION

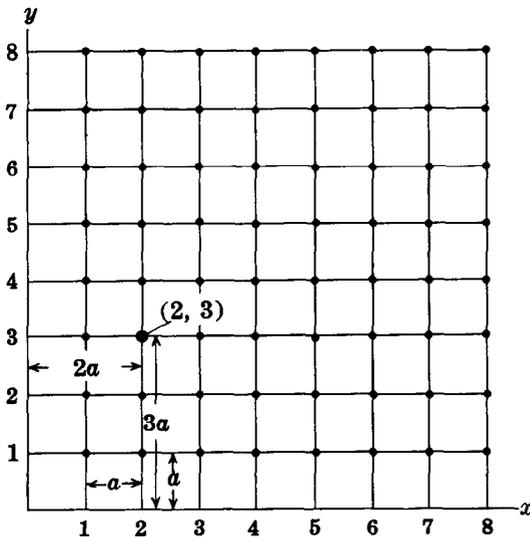
Again assuming $\Delta x = \Delta y = a$, which of the following would give the actual coordinate distances and time of measurement associated with the term $h_{i,j,n}$?

Turn to Section:

- $h_{i,j,n}$ = head at $x = i \cdot a, y = i \cdot \Delta t, \text{ time} = n \cdot \Delta t$ 9
- $h_{i,j,n}$ = head at $x = i \cdot \Delta x, y = i \cdot \Delta y, \text{ time} = n \cdot a$ 23
- $h_{i,j,n}$ = head at $x = i \cdot a, y = j \cdot a, \text{ time} = n \cdot \Delta t$ 10

5 □ □

Your answer, $h_{ia,ja}$, in Section 3 is not correct. You have used the distances from the two coordinate axes as subscripts. That is,



you have used ia , which is actually the x coordinate of the node, or its distance from the y axis, as the first subscript; and you have used ja , which is actually the y coordinate of the node, or its distance from the x axis, as the second subscript. The convention introduced in Section 3, however, does not have this form. If the finite-difference grid is superimposed on the x, y plane, as in the sketch, then the subscript associated with the point $x = 2a, y = 3a$ is simply 2, 3; the head at this point is designated $h_{2,3}$. If we number the lines of the grid in succession along each axis, starting with the axis as 0, we can obtain the subscript of a given node, or grid intersection, by looking at the numbers assigned to the two grid lines which intersect there; point 2, 3 is at the intersection of vertical line number 2 and horizontal line number 3.

Return to Section 3 and choose another answer.

6

Your answer in Section 25 is not correct. Your formulation for the calculation of the new value of $h_{i,j}$ in the first step is incorrect. The finite-difference equation which we developed stated that the value of $h_{i,j}$ should be the average of the values of h at the four surrounding nodes, that is

$$h_{i,j} = \frac{1}{4}(h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}).$$

The idea in the relaxation process is to compute a new value of $h_{i,j}$ as the average of the previous values of h at the four surrounding nodes. That is

$$h_{i,j}(\text{New Value}) = \frac{1}{4}(h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}) \text{ (Previous Values).}$$

When this calculation has been made, the idea is to compare the new value of $h_{i,j}$ with the previous value of $h_{i,j}$. If these two are very close, everywhere in the grid, there is no point in continuing the process further, since additional iterations will produce little additional change. The solution, in other words, has converged to values of h which satisfy the difference equation. In the second step, therefore, rather than setting $R_{i,j}$ equal to the average of the new and previous values of $h_{i,j}$ as in the answer you selected, $R_{i,j}$ should be set equal to the difference between $h_{i,j}$ (New Value) and $h_{i,j}$ (Previous Value). This difference may then be tested throughout the grid, and if it is sufficiently small at all points, the iteration process can be terminated.

Return to Section 25 and choose another answer.

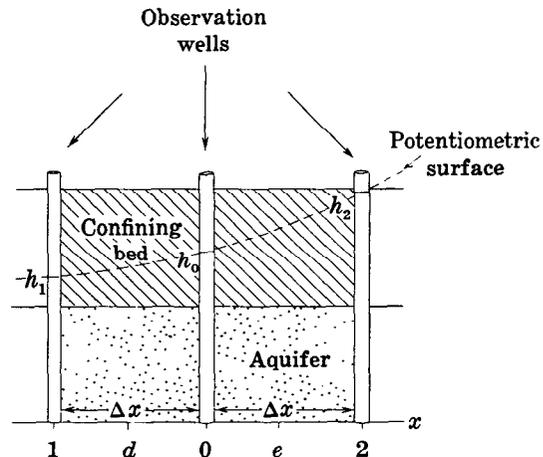
7

Your answer in Section 1,

$$\frac{h_1 - h_2}{\Delta x}$$

is not correct. In introducing the notion of a derivative, it is customary to begin with the finite-difference form—that is, to consider the finite change in h , Δh , occurring over a finite interval, Δx , along the x axis. The derivative notation, dh/dx , is then introduced to represent the value of the ratio $\Delta h/\Delta x$, as Δx becomes infinitesimal in size. Here, the idea is to move in the opposite direction. We started with the derivative, $\partial h/\partial x$, and we wish to approximate it by a ratio of finite differences. Moreover, we want an expression which applies at point d , midway between well 0 and well 1. The finite change in h occurring between these two wells is $h_0 - h_1$. The finite distance separating them is Δx .

Return to Section 1 and choose another answer.



8 □ □

Your answer in Section 10 is not correct. You have used the correct formulation for the forward-difference approximation to $\partial h / \partial t$ —that is,

$$\frac{\partial h}{\partial t} \approx \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t}$$

—but your approximation for $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$ is not correct. To obtain an approximation for $\partial^2 h / \partial x^2$, we move along the x axis, holding y constant. In this process i , the subscript denoting node position on the x axis will change, whereas j , the subscript denoting node position in the y direction, will remain unchanged. Our result will be

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{\frac{h_{i+1,j,n} - h_{i,j,n}}{a} - \frac{h_{i,j,n} - h_{i-1,j,n}}{a}}{a} = \frac{h_{i+1,j,n} + h_{i-1,j,n} - 2h_{i,j,n}}{a^2}$$

Similarly, in obtaining an approximation for $\partial^2 h / \partial y^2$, we move along the y axis, so that i remains fixed, while the y -subscript, j , varies. The result is

$$\frac{\partial^2 h}{\partial y^2} \approx \frac{\frac{h_{i,j+1,n} - h_{i,j,n}}{a} - \frac{h_{i,j,n} - h_{i,j-1,n}}{a}}{a} = \frac{h_{i,j+1,n} + h_{i,j-1,n} - 2h_{i,j,n}}{a^2}$$

Addition of these two expressions will give the correct approximation for $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$.

Return to Section 10 and choose another answer.

9 □ □

Your answer in Section 4 is not correct. The subscripts i, j, n tell us that head $h_{i,j,n}$ occurs at a certain node, i, j of the finite-difference grid on the x, y plane and at a certain point, n , of the finite-difference scale along the time axis. The coordinate values are found by multiplying the number of nodes along a given axis by the node spacing. Along the x axis the node i, j lies a distance $i \cdot a$ from the origin (i nodes, each with

spacing a). Along the time axis, the point n occurs at a time $n \cdot \Delta t$ (n time marks, each at a spacing Δt). The same procedure should be applied in determining the y coordinate, keeping in mind that there are j nodes along the y axis between the origin and point i, j , and that these nodes fall at a spacing a .

Return to Section 4 and choose another answer.

10 □ □

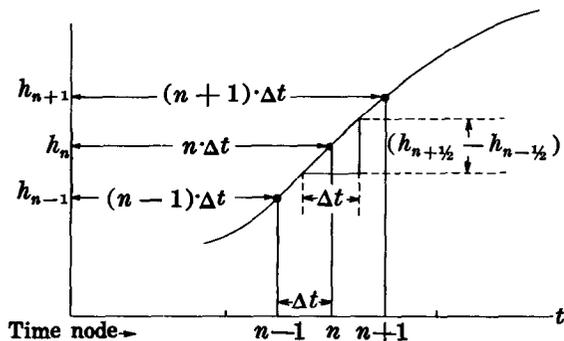
Your answer in Section 4 is correct. On each axis, x , y , and t , the value of the independent variable is found by multiplying the subscript, or node number, by the node spacing along the axis. Using the conventions we have adopted, therefore, the approximation

$$\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)_{n\Delta t} \approx \frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} - 4h_{i,j,n}}{a^2}$$

Now in order to simulate the differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t}$$

at the instant $t = n\Delta t$ we require in addition an approximation to $\partial h / \partial t$ at this instant.



The sketch shows a graph of h versus t in the vicinity of this time. A reasonable approximation to $\partial h / \partial t$ in the vicinity of the n th time mark would obviously be

$$\frac{\partial h}{\partial t} = \frac{h_{(n+1/2)} - h_{(n-1/2)}}{\Delta t}$$

to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

at the time $t = n\Delta t$, and at the point $x = ia$, $y = ja$ would be given by

In practical methods of computation, however, the approximations

$$\left(\frac{\partial h}{\partial t}\right)_{n\Delta t} \approx \frac{h_{n+1} - h_n}{\Delta t}$$

or

$$\left(\frac{\partial h}{\partial t}\right)_{n\Delta t} \approx \frac{h_n - h_{n-1}}{\Delta t}$$

are often found preferable. Here, we are simulating the derivative at $t = n\Delta t$ by, respectively, a "forward difference" taken between the times $n \cdot \Delta t$ and $(n+1) \cdot \Delta t$, and a "backward difference," taken between $(n-1) \cdot \Delta t$ and $n \cdot \Delta t$. The error involved will depend largely upon our choice of Δt , and can be reduced to tolerable limits by choosing Δt sufficiently small.

QUESTION

Using the forward-difference approximation to $\partial h / \partial t$ given above, which of the following results is obtained as a finite-difference simulation of the equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t}$$

at the point $x = ia$, $y = ja$, and at the time $t = n\Delta t$?

10 □ □ —Con.

Turn to Section:

$$\frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} - 4h_{i,j,n}}{a^2} = \frac{S}{T} \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t} \quad 16$$

$$\frac{h_{i-1,j-1,n} + h_{i+1,j+1,n} + h_{i+1,j-1,n} - h_{i-1,j+1,n} - 4h_{i,j,n}}{a^2} = \frac{S}{T} \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t} \quad 8$$

$$\frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} - 4h_{i,j,n}}{a^2} = \frac{S}{T} \frac{h_{i,j,n+\frac{1}{2}} - h_{i,j,n-\frac{1}{2}}}{\Delta t} \quad 19$$

11 □ □

Your answer in Section 16 is not correct. For the steady-state condition, $\partial h / \partial t = 0$; so, our equation,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t}$$

becomes simply

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

To obtain a finite-difference approximation to this equation, we need only take our finite-difference approximation to $(\partial^2 h / \partial x^2)$

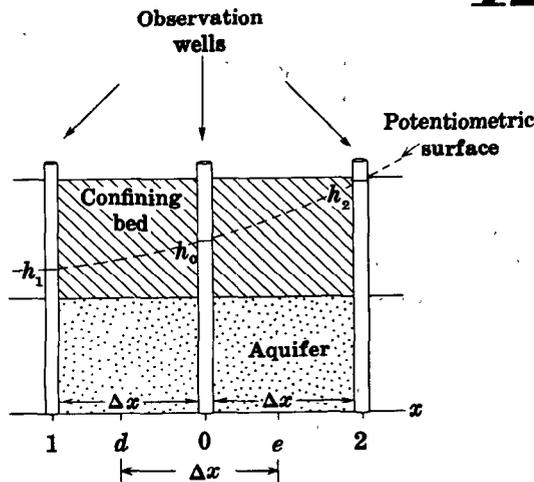
+ $(\partial^2 h / \partial y^2)$ and set it equal to zero. Our approximation to this sum, using the subscript notation associated with the finite-difference grid, was

$$\frac{h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j}}{a^2}$$

This expression can be set equal to zero, and the resulting equation multiplied through by the constant a^2 to obtain the finite-difference equation which we require.

Return to Section 16 and choose another answer.

12 □ □



Your answer in Section 1,

$$\left(\frac{\partial h}{\partial x} \right)_a \approx \frac{h_0 - h_1}{\Delta x}$$

is correct. Similarly the derivative at point e , midway between well 0 and well 2 is approximated by

(continued on next page)

12 —Con.

$$\left(\frac{\partial h}{\partial x}\right)_e \approx \frac{h_2 - h_0}{\Delta x}$$

QUESTION

Which of the following expressions gives a reasonable approximation for the second derivative, $\partial^2 h / \partial x^2$, at point 0—that is, at the location of the center well?

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_2 - h_1}{2\Delta x}$$

Turn to Section:

27

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_1 + h_2 - 2h_0}{(\Delta x)^2}$$

15

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{\frac{h_2 - h_0}{\Delta x} - \frac{h_0 - h_1}{\Delta x}}{2\Delta x}$$

22

13

Your answer in Section 16 is not correct. The finite-difference expression approximating

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

was

$$\frac{h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j}}{a^2}$$

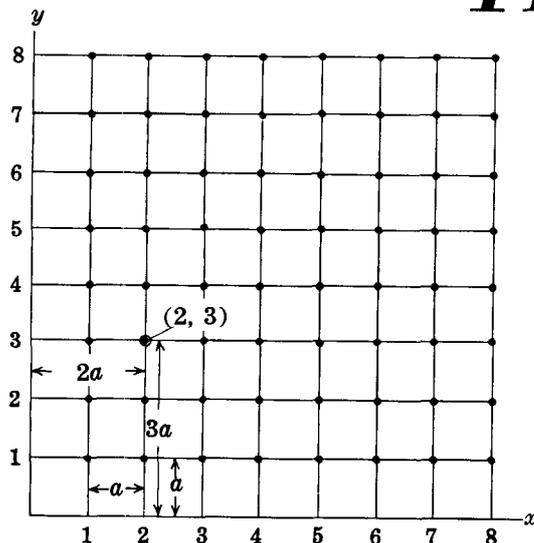
To approximate the equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

this finite-difference expression need only be equated to zero. The resulting equation can be multiplied through by the constant a^2 .

Return to Section 16 and choose another answer.

14

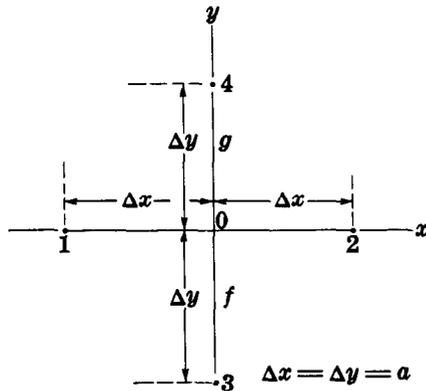


Your answer, $h_{j,i}$, in Section 3 is not correct. The sketch shows a diagram of the x, y plane, with the finite-difference grid superimposed upon it. Node 2, 3 is at a distance $2a$ from the y axis ($x=2a$) and a distance $3a$ from the x axis ($y=3a$). That is, the node having the coordinates $x=2a, y=3a$ is the node 2, 3; and the head at this node is designated $h_{2,3}$. The same rules apply for the node in the question of Section 3 which was at a distance $i \cdot a$ from the y axis and a distance $j \cdot a$ from the x axis. The coordinates of this node are $x=i \cdot a, y=j \cdot a$.

Return to Section 3 and choose another answer.

15

Map view



Your answer in Section 12,

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_1 + h_2 - 2h_0}{(\Delta x)^2},$$

is correct. If we were to consider, in addition, the wells 3 and 4 along a line parallel to the y axis (see figure), we would similarly have as an approximation for $\partial^2 h / \partial y^2$ at point 0,

$$\frac{\partial^2 h}{\partial y^2} \approx \frac{h_3 + h_4 - 2h_0}{(\Delta y)^2}.$$

QUESTION

If the spacing of the wells in the diagram is uniform—that is, if $\Delta x = \Delta y = a$ —which of the following expressions may be obtained for

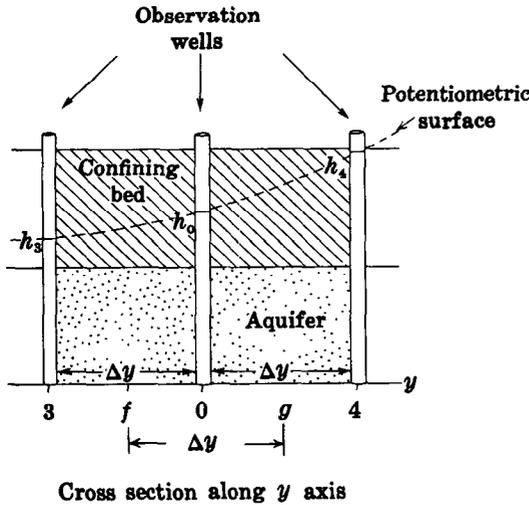
$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}?$$

Turn to Section:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_1 + h_2 + h_3 + h_4 - 4h_0}{a^2} \quad \mathbf{3}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_1 + h_2 + h_3 + h_4}{a^2} \quad \mathbf{28}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{(h_1 + h_2 - h_3 + h_4)}{a^2} \quad \mathbf{24}$$



16

Your answer in Section 10 is correct. Note that the equation which we have obtained is actually an algebraic equation, involving the terms $h_{i-1,j,n}$, $h_{i+1,j,n}$, $h_{i,j-1,n}$, $h_{i,j+1,n}$, $h_{i,j,n}$, and $h_{i,j,n+1}$; that is, we have simulated a differential equation by an algebraic equation. If the values of head are known at all nodes

of the x, y plane for some initial time, $t=0$, then the head value at each internal node for the succeeding time, $t=1 \cdot \Delta t$, can be obtained by applying the equation we have just obtained at the two times 0 and $1 \cdot \Delta t$ ($n=0$ and $n=1$). This would give

$$\frac{h_{i-1,j,0} + h_{i+1,j,0} + h_{i,j-1,0} + h_{i,j+1,0} - 4h_{i,j,0}}{a^2} = \frac{S}{T} \frac{h_{i,j,1} - h_{i,j,0}}{\Delta t}.$$

16 □ □ —Con.

This equation is applied in turn at each internal node of the plane and solved for $h_{i,j,1}$ at each point, using the appropriate values of h from the $t=0$ distribution. Additional conditions must be given from which head values at nodes along the boundaries of the x, y plane at the new time can be determined. When the head values are determined throughout the plane for the new time ($n=1$), the procedure may be repeated to determine head values at the next point on the time axis ($n=2$); and so on.

This is termed the explicit procedure of solution. It suffers from the shortcoming that if Δt is chosen too large, errors may be introduced which grow in size as the stepwise calculation proceeds, so that for large values of time the solution bears no relation to reality, even as an approximation. To circumvent this difficulty, other schemes of computation are often used, some involving the backward-difference approximation to $\partial h/\partial t$, and others involving entirely different simulations of the differential equation.

Many of these schemes of solution involve iterative techniques, in which the differences between members of an equation are successively reduced by numerical adjustment. These techniques are sometimes termed re-

laxation methods; they are of sufficient importance that it will be worthwhile to see how they operate, through a simple example.

Suppose we are dealing with a problem of two-dimensional steady-state ground-water flow. For a steady state situation, the term $\partial h/\partial t$ of our differential equation, and therefore the term

$$\frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t}$$

of our finite-difference equation, is zero. The differential equation is simply

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

QUESTION

Using the notation developed above, but dropping the third subscript since time is not involved, which of the following would represent a valid finite-difference approximation to this steady-state equation?

	Turn to Section:
$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j} = 0$	25
$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} + 4h_{i,j} = a^2$	11
$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} = \frac{4h_{i,j}}{a^2}$	13

17 □ □

Your answer in Section 25 is correct. If we were to "flow chart" the relaxation procedure for solution on a digital computer, we would have to incorporate these steps in some way.

Numerous other techniques exist for the numerical solution of the differential equations of flow. The efficiency of various methods, in terms of computational labor or machine time, varies widely depending upon the problem under study. Care must be exercised in selecting a method that is well suited to the problem, or unreasonable investments of time and effort may be required to obtain a solution.

In this discussion we have given only a brief indication of the way in which numerical methods may be applied in ground-water hydrology. Numerical analysis is a broad and complex field in itself. Interested readers will find an extensive literature dealing both with theory and with a wide range of applications. Examples of the use of numerical techniques in ground water may be found in the work of Prickett and Lonquist (1971), Stallman (1956), Remson, Appel, and Webster (1965), Pinder and Bredehoeft (1968), Rubin (1968), Bredehoeft and Pinder (1970), Freeze (1971), Prickett and Lonquist (1973), Trescott, Pinder, and

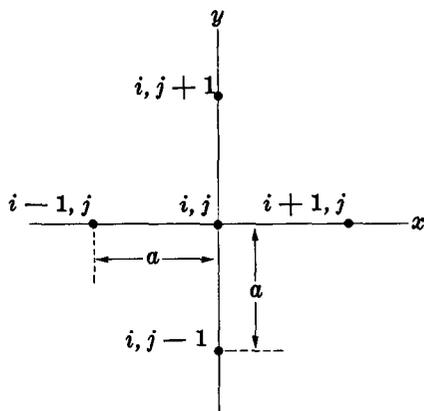
17

Jones (1970), Trescott, (1973), and many others. An excellent summary of numerical methods as applied in ground-water hydrology is given by Remson, Hornberger, and Molz (1970).

You have completed the programed instruction of Part VII. A discussion giving further details of some of the standard finite-difference techniques is presented in standard text format following Section 28.

18

Your answer in Section 2 is not correct. The sketch shows the five-well array which we used earlier to develop an approximation for $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$, but with the wells now redesignated according to the scheme of subscripts associated with our



finite-difference grid. The head at the central well is designated $h_{i,j}$ rather than h_0 ; the heads at the two wells along the x axis are $h_{i-1,j}$ and $h_{i+1,j}$, rather than h_1 and h_2 ; and the heads at the two wells along the y axis are designated $h_{i,j-1}$ and $h_{i,j+1}$, rather than h_3 and h_4 . Our previous expression for

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

was

$$\frac{h_1 + h_2 + h_3 + h_4 - 4h_0}{a^2}$$

The question only requires that this be translated into the notation associated with the finite-difference grid.

Return to Section 2 and choose another answer.

19

Your answer in Section 10 is not correct. Your approximation for $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$ is correct, but you have not used the forward-difference formulation to approximate $\partial h / \partial t$, as required by the question. The approximation which you have used,

$$\frac{\partial h}{\partial t} = \frac{h_{i,j,n+\frac{1}{2}} - h_{i,j,n-\frac{1}{2}}}{\Delta t},$$

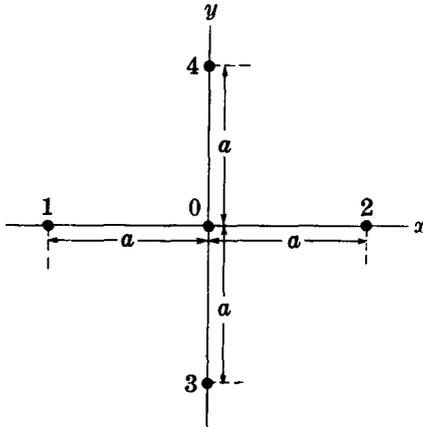
is normally a more accurate approximation to $\partial h / \partial t$ at i, j, n , than is the forward-difference formulation, since the difference is

taken symmetrically about the point at which $\partial h / \partial t$ is to be approximated. Unfortunately, however, it is not always as useful in the calculation of actual numerical solutions as is the forward-difference or backward-difference formulation. These formulations are unsymmetrical in the sense the difference is measured entirely to one side or the other of the time $t = n\Delta t$, which is the instant at which $\partial h / \partial t$ is to be approximated; but they are better suited to many techniques for computing solutions.

Return to Section 10 and choose another answer.

20 □ □

FIGURE A

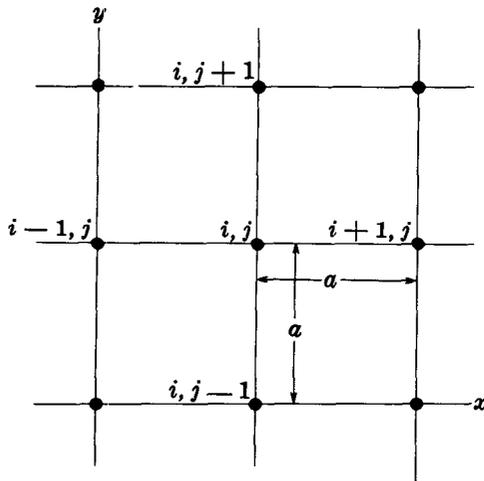


Your answer in Section 2 is not correct. The upper part of the figure shows the array which we used in developing our finite-difference approximation for $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$. The well at the center of the array was labeled 0; the surrounding wells were labeled as indicated. The expression we obtained for

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

was

$$\frac{h_1 + h_2 + h_3 + h_4 - 4h_0}{a^2}$$



Using the notation introduced for our finite-difference grid, shown in the lower part of the figure, the well at the center of the array would be denoted i, j ; the remaining wells would be designated: $i-1, j$; $i+1, j$; $i, j-1$; and $i, j+1$, as shown. It is simply a matter of substituting these designations for the designations, 0, 1, 2, 3, and 4 used in our earlier development.

Return to Section 2 and choose another answer.

21 □ □

Your answer in Section 25 is not correct. Your initial step, giving the formulation for computing the new value of $h_{i,j}$ using the previous values of $h_{i-1,j}$, $h_{i+1,j}$, $h_{i,j-1}$, and $h_{i,j+1}$, is correct. However, your second step is not correct. The idea is to continue the process until the difference between the previous value of $h_{i,j}$ and the new value of $h_{i,j}$

becomes very small everywhere in the grid. Thus $R_{i,j}$ should represent the difference between $h_{i,j}$ (New Value) and $h_{i,j}$ (Previous Value); and the process should be continued until $|R_{i,j}|$ is negligible throughout the grid.

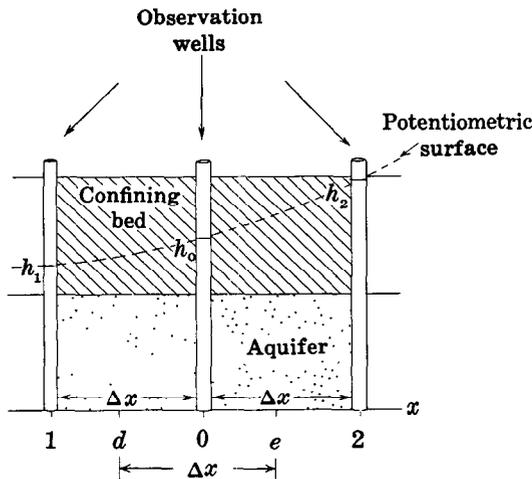
Return to Section 25 and choose another answer.

22 □ □

Your answer in Section 12,

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{\frac{h_2 - h_0}{\Delta x} - \frac{h_0 - h_1}{\Delta x}}{2\Delta x},$$

is not correct. The numerator in your answer gives the difference between two terms: $(h_2 - h_0)/\Delta x$, which approximates $\partial h/\partial x$ at point e ; and $(h_0 - h_1)/\Delta x$, which approximates $\partial h/\partial x$ at point d .



The numerator thus represents the difference

$$\left(\frac{\partial h}{\partial x}\right)_e - \left(\frac{\partial h}{\partial x}\right)_d,$$

that is, it approximates the change in $\partial h/\partial x$ between point d and e . Thus if it were divided by Δx , the interval between points d and e , we would have an approximation to

$$\frac{\partial\left(\frac{\partial h}{\partial x}\right)}{\partial x},$$

that is, to $\partial^2 h/\partial x^2$ at the midpoint, 0, of the interval between d and e . In the answer which you selected, however, the quantity

$$\frac{h_2 - h_0}{\Delta x} - \frac{h_0 - h_1}{\Delta x}$$

is divided by $2\Delta x$, rather than by Δx .

Return to Section 12 and choose another answer.

23 □ □

Your answer in Section 4 is not correct. The coordinate of a point, in space or time, is found by multiplying the number of nodes between the origin and the point in question, along the appropriate axis, by the node spacing along that axis. Thus the x coordinate of a node i, j, n , is $x = i \cdot a$, since there are i nodes along the x axis from the origin

to i, j , and the node spacing is a . The same procedure may be applied along the y and t axes, keeping in mind that the node spacing along the y axis is a , while that along the time axis is Δt .

Return to Section 4 and choose another answer.

24

Your answer in Section 15,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{(h_1 + h_2) - (h_3 + h_4)}{a^2}$$

is not correct. The approximate expression which we obtained for $\partial^2 h / \partial x^2$ was

$$\frac{h_1 + h_2 - 2h_0}{(\Delta x)^2}$$

or, since we have taken $\Delta x = a$,

$$\frac{h_1 + h_2 - 2h_0}{a^2}$$

The expression given in Section 15 for $\partial^2 h / \partial y^2$ was

$$\frac{h_3 + h_4 - 2h_0}{(\Delta y)^2}$$

or again, since we have taken $\Delta y = a$,

$$\frac{h_3 + h_4 - 2h_0}{a^2}$$

These two expressions need only be added algebraically to obtain an approximation for

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

Return to Section 15 and choose another answer.

25

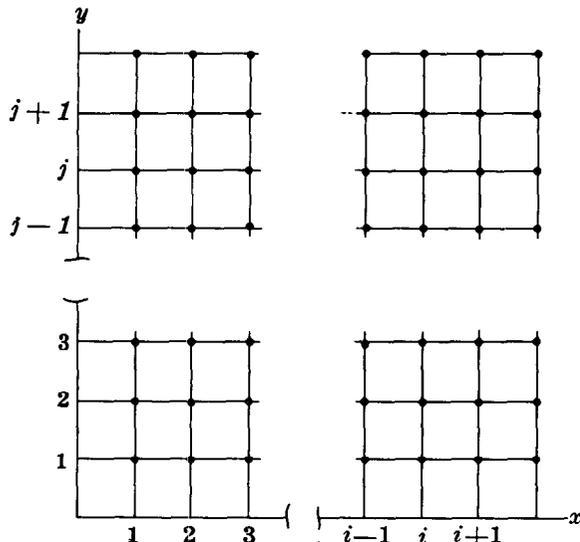
Your answer in Section 16

$$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j} = 0$$

is correct. To solve this by an iteration technique we rewrite the equation in the form

$$h_{i,j} = \frac{1}{4}(h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}),$$

and we divide the x, y plane into a grid as shown in the sketch, with the grid intersections forming the nodes at which we will compute values of h . In the form in which



we have written it, it is easy to see that what our equation actually says is that the head at each node must be the average of the heads at the four adjacent nodes. We begin by entering known values of head along the boundaries of the grid—that is, by applying the boundary conditions. We then insert assumed values of h at each interior grid point. These initial values of h may be anything we wish, although a great deal of work can be saved if we can choose them in a way that roughly approximates the final head distribution. We then move through the grid, in any order or direction, and at each interior node cross out the value of head, writing in its place the average of the head values at the four adjacent nodes. At each node we note not only the new value of h , but the *change* in h , from the initial value, resulting from the calculation. When we have completely traversed the grid, we start again, and proceed through the grid in the same way, replacing each h value by the average of the heads at the four adjacent nodes, and noting the change in h that this causes. After a number of repetitions we will find that the change in h caused by each new calculation becomes very small—in other words, that the value of head at each point is already essentially equal to the aver-

25 □ □ —Con.

age of those at the four neighboring points, so that inserting this average in place of h produces little or no additional change. At this point our head distribution represents an approximate solution to our difference equation and thus to the differential equation which the difference equation simulates.

The process just described, as noted earlier, is an example of a relaxation technique.

In general, since the head at each node is used in calculating the head at each of the four surrounding nodes, several complete traverses of the grid may be required before the changes in head are everywhere sufficiently small. This method can readily be used in hand calculation; it is also well adapted to solution by digital computer.

QUESTION

Which of the following would you choose as a "shorthand" description of the method of calculation described above?

Turn to Section:

$$h_{i,j} \text{ (New Value)} = \frac{1}{4} (h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}) \text{ (Previous Values)}$$

$$R_{i,j} = h_{i,j} \text{ (New Value)} - h_{i,j} \text{ (Previous Value)}$$

Continue calculation until $|R_{i,j}| \approx 0$ for all points in grid. 17

$$h_{i,j} \text{ (New Value)} = \frac{1}{4} (h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}) \text{ (Previous Values)}$$

$$R_{i,j} = h_{i,j} \text{ (New Value)}$$

Continue calculation until $|R_{i,j}| \approx 0$ for all points in grid. 21

$$h_{i,j} \text{ (New Value)} = \frac{1}{4} (h_{i+1,j} - h_{i-1,j} + h_{i,j+1} - h_{i,j-1}) \text{ (Previous Values)}$$

$$R_{i,j} = \frac{h_{i,j} \text{ (New Value)} + h_{i,j} \text{ (Previous Value)}}{2}$$

Continue calculation until $|R_{i,j}| \approx 0$ for all points in grid. 6

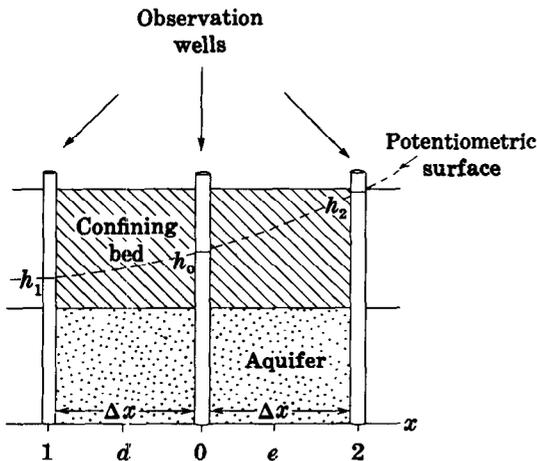
26 □ □

Your answer in Section 1,

$$\left(\frac{\partial h}{\partial x} \right)_d \approx \frac{h_2 - h_1}{2\Delta x}$$

is not correct. This answer would be a reasonable approximation for the derivative at point 0, in the center of the array, because it gives the ratio of a change in h , $h_2 - h_1$, to the corresponding change in distance, $2\Delta x$,

26 □ □ —Con.



over an interval which is centered at 0. For the derivative at point d , however, midway between well 1 and well 0, we can do a little better. The change in h over an interval centered at d is simply $h_0 - h_1$, and the corresponding interval of distance is simply Δx .

Return to Section 1 and choose another answer.

27 □ □

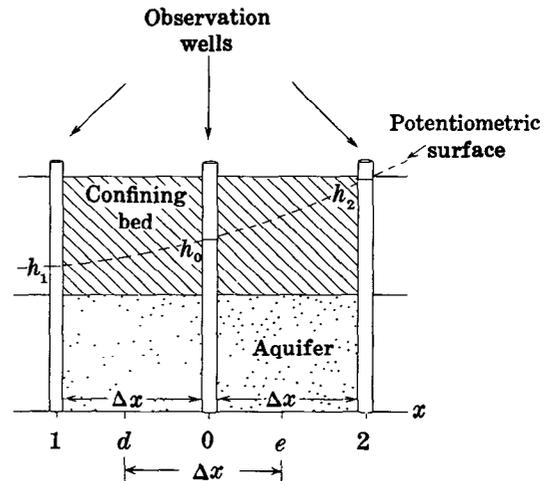
Your answer in Section 12,

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_2 - h_1}{2\Delta x}$$

is not correct. $h_2 - h_1$ gives the change in h between points 1 and 2, and $2\Delta x$ gives the distance between these points. Thus the term $(h_2 - h_1)/2\Delta x$ is an approximation to the first derivative, $\partial h/\partial x$, at the midpoint of the distance interval—that is, at point 0. The question however, asked for a term approximating the second derivative, $\partial^2 h/\partial x^2$, at this point. The second derivative is actually the derivative of the first derivative; that is

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial \left(\frac{\partial h}{\partial x} \right)}{\partial x}$$

To obtain a finite-difference expression for this term, we must consider the change in the first derivative, $\partial h/\partial x$, between two points, and must divide this change in $\partial h/\partial x$ by the distance separating these two points. We have seen that $\partial h/\partial x$ at point d , midway between wells 1 and 0, can be approximated by the expression $(h_0 - h_1)/\Delta x$; and that $\partial h/\partial x$ at point e , midway between



wells 0 and 2 can be approximated by the term $(h_2 - h_0)/\Delta x$. Points d and e are themselves separated by a distance Δx , and point 0 is at the midpoint of this interval. Thus if we subtract our approximate expression for $\partial h/\partial x$ at d , from that for $\partial h/\partial x$ at e , and divide the result by the interval between d and e , Δx , we should obtain an expression for $\partial^2 h/\partial x^2$ at point 0.

Return to Section 12 and choose another answer.

28

Your answer in Section 15,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_1 + h_2 + h_3 + h_4}{a^2}$$

is not correct. The term $-2h_0$ appeared in the numerator of both of our approximate

expressions —that for $\partial^2 h / \partial x^2$ and that for $\partial^2 h / \partial y^2$. When we add these two expressions to obtain an approximation for $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$, these terms in h_0 do not drop out.

Return to Section 15 and choose another answer.

Techniques of Finite-Difference Solution of the Ground-Water-Flow Equation

Certain techniques of numerical solution which are commonly used in ground-water modeling are described in the following discussion. No attempt has been made to discuss such topics as stability or rate of convergence in theoretical terms; the reader is referred to the paper by Peaceman and Rachford (1955) for discussion of these subjects. Similarly, no attempt has been made to give the details of the programming procedure. The paper by Prickett and Lonquist (1971) analyzes some typical programs and in addition provides an excellent summary of the hydrologic and mathematical foundations of digital modeling; the paper by Trescott (1973) describes a versatile program for areal aquifer simulation. The discussion presented here is limited to a description of some of the common techniques of approximation and calculation.

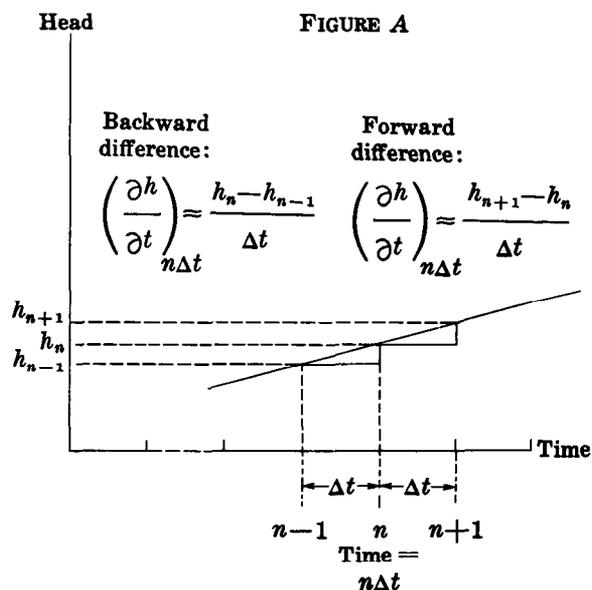
In Section 10 of Part VII we introduced two methods of approximating the time derivative in finite-difference simulations of the ground-water equation. One of these was termed the forward-difference approximation, and one the backward-difference approximation. Figure A shows a plot of head versus time which we may use to review these approximations. The time axis is divided into intervals of length Δt . The head at the end of the n th interval is termed h_n ; that at the end of the preceding interval is termed h_{n-1} ; and that at the end of the subsequent interval is termed h_{n+1} . We wish to approximate $\partial h/\partial t$ at the end of the n th interval, that is, at the time $n\Delta t$. If we utilize the head difference over the subsequent time interval, we employ the forward-difference approximation to the time derivative; if we utilize the head difference over the preced-

ing interval, we employ the backward-difference approximation. The forward-difference approximation is given by

$$\left(\frac{\partial h}{\partial t}\right)_{n\Delta t} \approx \frac{h_{n+1} - h_n}{\Delta t} \quad (1)$$

Where $(\partial h/\partial t)_{n\Delta t}$ represents the derivative at time $n\Delta t$. The backward-difference approximation is given by

$$\left(\frac{\partial h}{\partial t}\right)_{n\Delta t} \approx \frac{h_n - h_{n-1}}{\Delta t} \quad (2)$$



Forward-difference simulation: Explicit solution

The ground-water-flow equation, as it was given in Part V for two-dimensional flow, is

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad (3)$$

where S represents storage coefficient and T transmissivity. In order to simulate this equation using either the forward-difference or backward-difference formulation, we would first write an approximate expression for the term

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

at the time $n\Delta t$ —that is, at point n on the time axis of figure A. Thus the forward-difference simulation is characterized by the fact that we approximate $\partial h/\partial t$ over a time interval which follows the time at which we approximate $(\partial^2 h/\partial x^2) + (\partial^2 h/\partial y^2)$, whereas the backward-difference simulation is characterized by the fact that we approximate $\partial h/\partial t$ over the time interval which precedes the time at which we approximate $(\partial^2 h/\partial x^2) + (\partial^2 h/\partial y^2)$. In the question of Section 10, Part VII, we obtained the following forward-difference simulation to equation 3:

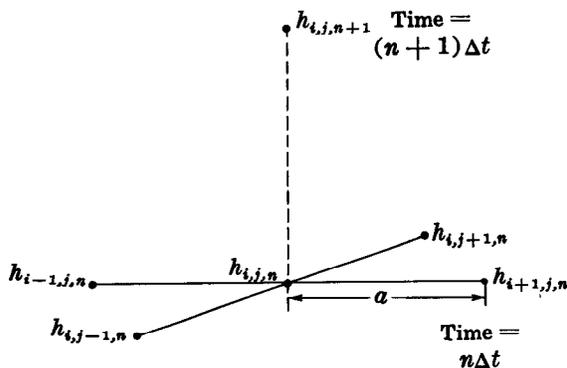
$$\frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} - 4h_{i,j,n}}{a^2} = \frac{S}{T} \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t} \quad (4)$$

where a is the node spacing, S is the storage coefficient, and T is the transmissivity. We wish to know the new value of head at the time $(n+1)\Delta t$ for the point i, j . Figure B shows the computation stencil for this simulation; the head at node i, j at the time $(n+1)\Delta t$ depends on the head in a five-node array at the preceding time, $n\Delta t$. The five values of h at the time $n\Delta t$ are all known. We need only to rearrange the equation, solving for $h_{i,j,n+1}$, and to insert the known

values of $h_{i-1,j,n}$, $h_{i+1,j,n}$, $h_{i,j-1,n}$, $h_{i,j+1,n}$, and $h_{i,j,n}$. There is no need to use simultaneous equations; the head at each node is computed explicitly, using the head at that node and the four neighboring nodes from the preceding time. The sequence in which we move through the x, y plane, calculating new values of head, is immaterial. The solution at one point does not require information on the surrounding points for the same time—only for the preceding time. For all these reasons, the forward-difference technique is computationally simpler than the backward-difference technique.

However, as we noted earlier, the forward-difference method does suffer from a serious drawback. Unless the ratio $\Delta t/a^2$ is kept sufficiently small, errors which grow in magnitude with each step of the calculation may appear in the result. More exactly, let us suppose that an error of some sort does arise, for whatever reason, at a certain node at a particular time step. Unless the ratio $\Delta t/a^2$ is sufficiently small, this error will increase in magnitude at each succeeding time

FIGURE B



step in the calculation until eventually the error completely dominates the solution. The term "error," as used here, refers to any difference between the computed head at a node i, j and time $n\Delta t$, and the actual value of head—that is, the value which would be given by the exact solution to the differential equation at that point and time. Such errors are inevitable in the normal application of finite-difference methods; they generally ap-

pear throughout the mesh in the first steps of the calculation. If the restriction on $\Delta t/a^2$ is satisfied, these errors will tend to die out as the computation sequence continues; the solution is then said to be stable. If the restriction is not satisfied, the errors will grow with each succeeding time step and will eventually destroy any significance which the solution might have; in this case, the solution is said to be unstable.

Backward-difference simulation: Solution by iteration

Because of this limitation in the forward-difference approach, attention has been given to a variety of alternative methods. One of these is simulation of the differential

equation 3 through use of the backward-difference approximation to the time derivative as given in equation 2. The resulting finite-difference equation is

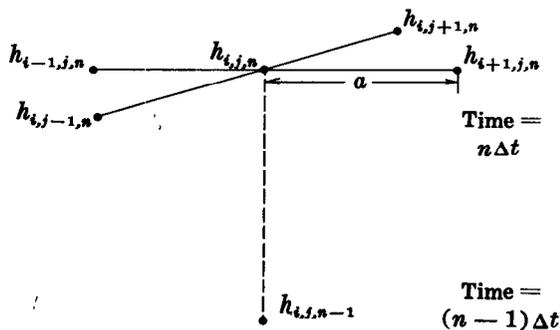
$$\frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} - 4h_{i,j,n}}{a^2} = \frac{S}{T} \frac{h_{i,j,n} - h_{i,j,n-1}}{\Delta t} \quad (5)$$

Figure C shows a diagram of the computation stencil for equation 5. The time derivative is simulated over an interval which precedes the time at which $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$ is simulated; the equation incorporates five unknown values of head, corresponding to the time $n\Delta t$, and only one known value of head, corresponding to the time $(n-1)\Delta t$. Clearly we cannot obtain an

explicit solution to a single equation of the form of equation 5, the way we could to a single equation of the form of equation 4. We can, however, write an equation of the form of equation 5 for each node in the x, y plane; then since there is one unknown value of head (for time $t = n\Delta t$) at each node in the plane, we will have a system in which the total number of equations is equal to the total number of unknowns. We should therefore be able to solve the entire set as a system of simultaneous equations, obtaining the new value of $h_{i,j,n}$ at each node. The only drawback to this approach is that a great deal of work may be involved in solving the set of simultaneous equations; offsetting this drawback is the advantage that the technique is stable regardless of the size of the time step—that is, that errors tend to diminish rather than to increase as the computation proceeds, regardless of the size of Δt relative to a^2 .

The work required in utilizing the back-

FIGURE C



ward-difference technique depends upon the size of the problem—that is, upon the number of equations in the simultaneous set. If this number becomes large, as it does in most ground-water problems, the work entailed becomes very great, particularly when the standard direct methods of solving simultaneous equations are used. For this reason it is worthwhile to look for efficient methods of solving these sets of equations; and it turns out that iteration or relaxation—the process described in Section 25 of Part VII, in connection with solution of the steady-state equation—provides us with a reasonably efficient approach.

The equation that we were trying to solve by iteration in Section 25 of Part VII rewritten here using the i, j subscript notation, is

$$\frac{1}{4}(h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}) = h_{i,j}. \quad (6)$$

This equation states that the head at the node i, j should be the average of the heads at the four surrounding nodes. No time subscripts are involved, since we are dealing with a steady-state situation. Our method is simply to move through the x, y plane, replacing the head at each node by the average of the heads at the four surrounding nodes. This process is continued until the head changes become negligible—that is, until the head at each node remains essentially unchanged after each traverse through the plane, indicating that equation 6 is satisfied throughout the plane.

In applying iteration to our nonequilibrium problem, the idea is to carry out a similar series of traverses of the x, y plane at every time step, using equation 5 rather than equation 6 as the basis of the calculation at each node. Thus to compute heads for the time $n\Delta t$ we would rearrange equation 5 as follows

$$h_{i,j,n} = \left(\frac{1}{\frac{4}{a^2} + \frac{S}{T\Delta t}} \right) \left\{ \frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n}}{a^2} + \frac{S}{T\Delta t} h_{i,j,n-1} \right\}. \quad (7)$$

We can envision an x, y plane for the time $n\Delta t$, initially containing specified values of $h_{i,j,n}$ at a few nodes, corresponding to the boundary conditions, and trial values of $h_{i,j,n}$ at the remaining nodes. We write an equation of the form of equation 7 for every node not controlled by a boundary condition; and we write equations expressing the boundary conditions for the nodes at which these conditions apply. In equation 7, the value of $h_{i,j,n}$ is expressed in terms of the head at the four surrounding nodes for the same time, and the head at the same node for the preceding time. In solving the set of equations for values of $h_{i,j,n}$ the values of $h_{i,j,n-1}$ actually constitute known or constant

terms, determined in the preceding step of the operation. Thus equation 7 relates the head at each node to the head at the four surrounding nodes, in terms of a set of constants or known quantities. The equation is a little more cumbersome than equation 6 in that instead of multiplying the sum of the heads at the surrounding nodes by $\frac{1}{4}$, we must now multiply by the term

$$\frac{1}{\left(\frac{4}{a^2} + \frac{S}{T\Delta t} \right) \cdot a^2}$$

and we must add the known term

$$\frac{\frac{S}{T\Delta t}}{\frac{4}{a^2} + \frac{S}{T\Delta t}} \cdot h_{i,j,n-1}$$

on the right side. These changes, however, do not make the equation appreciably more difficult to solve. We can still use the process of iteration; that is, we can move through the x, y plane, replacing each original trial value of $h_{i,j,n}$ by a new value, calculated from the four surrounding values by equation 7. At each node we note the difference between the new value of $h_{i,j,n}$ which we have calculated, and the trial value with which we started. If this difference turns out to be negligible at every node, we may conclude that our starting values already satisfied equation 7 and that further computation of new values is pointless. More commonly, however, we will note a measurable change in the value of h at each node, indicating that the initial values did not satisfy equation 7, and that the iteration procedure is producing an adjustment toward new values which will satisfy the equation. In this case we traverse the x, y plane again, repeating the procedure; each value of $h_{i,j,n}$ which we calculated in the first step (or iteration) is replaced by a new value calculated from the heads at the four surrounding nodes by equation 7. Again the difference between the new value and the preceding value at each node is recorded; and a test is made to see whether this difference is small enough to indicate that the new array of head values approximately satisfies equation 7. The process is continued until the difference between newly computed and preceding values is negligible throughout the array, indicating that equation 7 is essentially satisfied at all points.

The technique described above is often referred to as the Gauss-Seidel method; it is basically the same procedure that was applied in Section 25 of Part VII to the steady-state problem. It is an example of a relaxation technique—a method of computation in which the differences between the two sides

of an equation are successively reduced by numerical adjustment, until eventually the equation is satisfied. There are a number of varieties of relaxation techniques in use, differing from one another in the order or sequence in which the x, y plane is traversed in the calculation and in certain other respects.

It has been found that the number of calculations required to solve the set of finite-difference equations can frequently be reduced by the inclusion of certain “artificial” terms in these equations. These terms normally take the form

$$\lambda (h_{i,j,n}^{m+1} - h_{i,j,n}^m).$$

The superscripts m and $m+1$ indicate levels of iteration; that is, $h_{i,j,n}^m$ represents the value of $h_{i,j,n}$ after m traverses of the x, y plane in the iteration process, and $h_{i,j,n}^{m+1}$ represents the value of $h_{i,j,n}$ obtained in the next following calculation, after $m+1$ traverses. λ is termed an “iteration parameter”; it is a coefficient which, either on the basis of practical experience or theoretical analysis, has been shown to produce faster rates of solution. As the iteration process approaches its goal at each time step, the difference between the value of $h_{i,j,n}$ obtained in one iteration and that obtained in the next iteration becomes negligible—that is, the term $(h_{i,j,n}^{m+1} - h_{i,j,n}^m)$ approaches zero, so that the difference equation appears essentially in its original form, without the iteration parameter term; and the solution which is obtained thus applies to the original equation. In some cases, λ is given a sequence of different values in successive iterations, rather than a single constant value. Again, the particular sequence of values is chosen, either through theoretical analysis or through practical experience, in such a way as to produce the most rapid solution. When an iteration parameter or sequence of iteration parameters is utilized, the relaxation process is termed “successive overrelaxation” and is frequently designated by the initials *SOR*. Discussions of this technique are given by Forsythe and Wasow (1960) and many others.

Alternating-direction implicit procedure

The work required to obtain a solution by relaxation techniques is frequently tedious, particularly for a problem of large dimensions. For this reason, a great deal of effort has gone into the development of alternative approaches. Peaceman and Rachford (1955) proposed a technique of computation which has received wide use in a variety of forms. The name "alternating direction" has been applied to the general procedures of calculation which they proposed.

To simplify our discussion of their techniques we will introduce some new notation. We saw in Sections 12 and 15 of Part VII that an approximation to $\partial^2 h / \partial x^2$ is given by the term

$$\frac{h_1 + h_2 - 2h_0}{(\Delta x)^2};$$

or, in terms of our subscript notation,

$$\frac{h_{i-1,j} + h_{i+1,j} - 2h_{i,j}}{(\Delta x)^2}.$$

In the discussion which follows, we will let the symbol $\Delta_{xx}h$ represent this approximation to $\partial^2 h / \partial x^2$. That is, we say

$$\frac{\partial^2 h}{\partial x^2} \approx \Delta_{xx}h = \frac{h_{i-1,j} + h_{i+1,j} - 2h_{i,j}}{(\Delta x)^2}. \quad (8)$$

In addition, we will use a subscript to indicate the time at which the approximation is taken. For example, $(\Delta_{xx}h)_n$ will indicate an approximation to the second derivative at the time $n\Delta t$, or specifically

$$(\Delta_{xx}h)_n = \frac{h_{i-1,j,n} + h_{i+1,j,n} - 2h_{i,j,n}}{(\Delta x)^2}. \quad (9)$$

$(\Delta_{xx}h)_{n-1}$ will represent an approximation to the second derivative at time $(n-1)\Delta t$, and so on. Similarly, we will use the notation $\Delta_{yy}h$ to represent our approximation to $\partial^2 h / \partial y^2$, that is,

$$\frac{\partial^2 h}{\partial y^2} \approx \Delta_{yy}h = \frac{h_{i,j-1} + h_{i,j+1} - 2h_{i,j}}{(\Delta y)^2} \quad (10)$$

and again $(\Delta_{yy}h)_n$ will represent our approximation to $\partial^2 h / \partial y^2$ at the time $n\Delta t$, that is

$$(\Delta_{yy}h)_n = \frac{h_{i,j-1,n} + h_{i,j+1,n} - 2h_{i,j,n}}{(\Delta y)^2} \quad (11)$$

and so on.

Using this notation, our forward-difference approximation to the equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad (3)$$

as given in equation 4, would be rewritten

$$(\Delta_{xx}h)_n + (\Delta_{yy}h)_n = \frac{S}{T} \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t}. \quad (12)$$

In this formulation, $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$ are simulated at the beginning of the time interval over which $\partial h / \partial t$ is simulated.

Again using the notation introduced above, our backward-difference approximation to equation 3, as given in equation 5, would be rewritten

$$(\Delta_{xx}h)_n + (\Delta_{yy}h)_n = \frac{S}{T} \frac{h_{i,j,n} - h_{i,j,n-1}}{\Delta t}. \quad (13)$$

In this formulation, $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$ are simulated at the time $n\Delta t$, while $\partial h / \partial t$ is simulated over the time interval between $(n-1)\Delta t$ and $n\Delta t$; thus both $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$ are approximated at the end of the time interval over which $\partial h / \partial t$ is approximated.

In the form in which it was originally proposed, Peaceman and Rachford's technique is usually termed the alternating-direction implicit procedure. In this form, the

simulation utilizes two equations, applicable over two successive time intervals. In the first equation, $\partial^2 h / \partial x^2$ is simulated at the beginning of a time interval, and $\partial^2 h / \partial y^2$ at the end of that interval; $\partial h / \partial t$ is simulated using the change in head occurring over the interval. The second equation applies over the immediately following time interval; here the order is reversed— $\partial^2 h / \partial y^2$ is simulated at the beginning of the time interval, $\partial^2 h / \partial x^2$ is simulated at the end, and again $\partial h / \partial t$ is simulated using the head difference occurring over the interval.

Using the notation introduced above, this simulation may be represented by the following equation pair

$$(\Delta_{xx}h)_{n-1} + (\Delta_{yy}h)_n = \frac{S}{T} \cdot \frac{h_{i,j,n} - h_{i,j,n-1}}{\Delta t} \quad (14)$$

$$(\Delta_{yy}h)_n + (\Delta_{xx}h)_{n+1} = \frac{S}{T} \cdot \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t} \quad (15)$$

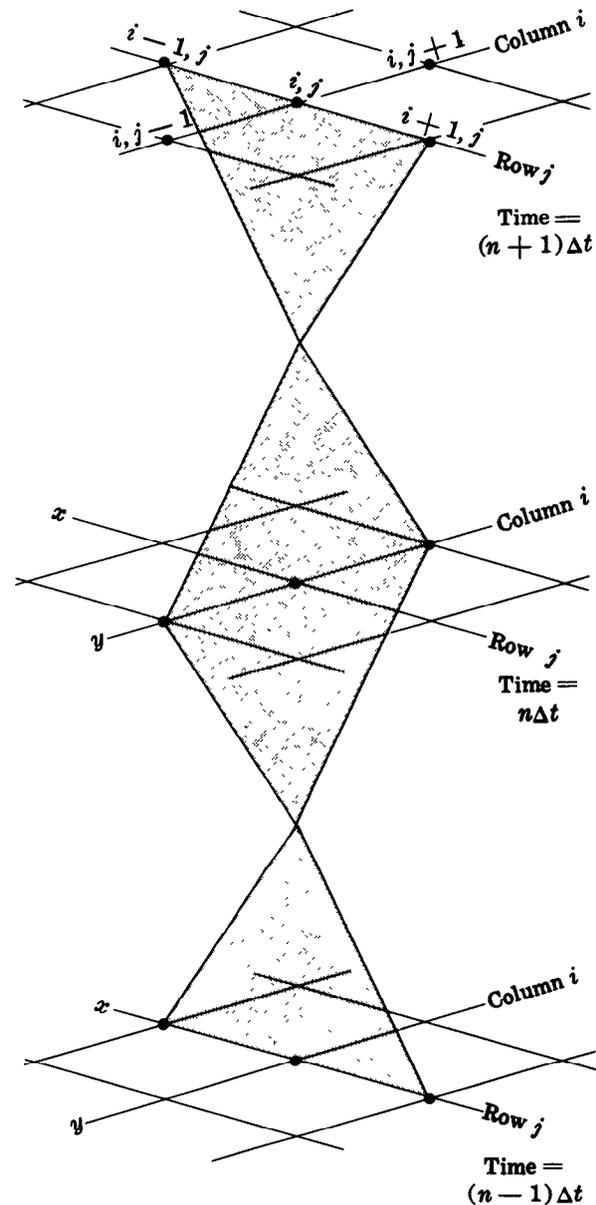
For the first time interval, $\partial^2 h / \partial x^2$ is simulated at $(n-1)\Delta t$; $\partial^2 h / \partial y^2$ is simulated at $n\Delta t$; and $\partial h / \partial t$ is simulated by the change in $h_{i,j}$ between $(n-1)\Delta t$ and $n\Delta t$. For the second time interval $\partial^2 h / \partial y^2$ is simulated at $n\Delta t$; $\partial^2 h / \partial x^2$ is simulated at $(n+1)\Delta t$; and $\partial h / \partial t$ is simulated by the change in $h_{i,j}$ between $n\Delta t$ and $(n+1)\Delta t$.

Figure D illustrates the form of this simulation. It may be recalled from Section 3 that lines parallel to the x -axis in the finite-difference grid are termed rows and that lines parallel to the y -axis are termed columns. As shown in figure D, then, three values of h are taken along row j at time $(n-1)\Delta t$ to simulate $\partial^2 h / \partial x^2$, while at the time $n\Delta t$ three values of h are taken along column i to simulate $\partial^2 h / \partial y^2$. The time derivative is simulated using the difference between the central h values at these two times. For the succeeding time interval, the three values of h along column i are taken first to simulate $\partial^2 h / \partial y^2$ at time $n\Delta t$; while at the time $(n+1)\Delta t$, three values of h are taken along row j to simulate $\partial^2 h / \partial x^2$. Again the time derivative is simulated using the

difference between the central h values.

The forward-difference and backward-difference techniques are characterized by symmetry in their simulation of the expression $(\partial^2 h / \partial x^2) + (\partial^2 h / \partial y^2)$. Both terms of this expression are simulated at the same time, using a five-node array centered about a single value of head, $h_{i,j,n}$. However, the

FIGURE D



simulation of $\partial h/\partial t$ in these formulations is asymmetrical, in the sense that it is not centered in time about $h_{i,j,n}$ but extends forward or backward from the time $n\Delta t$. In either case, however, if we allow Δt to become very small, the effects of this asymmetry die out; the approximation then approaches more and more closely the value of $\partial h/\partial t$ at the time $n\Delta t$. In the alternating-direction implicit procedure, by contrast, $\partial^2 h/\partial x^2$ and $\partial^2 h/\partial y^2$ are not simulated at the same time, and in this sense the simulation of $(\partial^2 h/\partial x^2) + (\partial^2 h/\partial y^2)$ cannot be termed symmetrical. It is again helpful, however, to visualize what will happen if Δt is allowed to become very small, so that the times $(n-1)\Delta t$ and $n\Delta t$ at which the individual simulations occur, fall more and more closely together. In this case, $(\Delta_x h)_{n-1}$ should begin to approximate the value of $\partial^2 h/\partial x^2$ at $(n-1/2)\Delta t$, while $(\Delta_y h)_n$ should begin to approximate the value of $\partial^2 h/\partial y^2$ at $(n-1/2)\Delta t$. In this sense, then, the expression

$$(\Delta_x h)_{n-1} + (\Delta_y h)_n$$

can be considered an approximation to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

at the time $(n-1/2)\Delta t$. The simulation of $\partial h/\partial t$ is symmetrical with respect to this time, since it utilizes the head difference $h_n - h_{n-1}$. Thus even though a certain asymmetry exists in the expression by which $(\partial^2 h/\partial x^2) + (\partial^2 h/\partial y^2)$ is approximated in the alternating-direction technique, it can be argued that there is symmetry with respect to time in the simulation of $\partial h/\partial t$. Moreover, we may expect intuitively that if an error is generated by the fact that we simulate $\partial^2 h/\partial x^2$ prior to $\partial^2 h/\partial y^2$ during one time interval, some sort of compensating error should be generated during the following time interval, when we simulate $\partial^2 h/\partial y^2$ prior to $\partial^2 h/\partial x^2$; and in fact it turns out that this alternation in the order of simulation is essential to the stability of the method. If the order of simulation is reversed in

this way, then regardless of the size of the time step, the calculation will not be affected by errors which grow at each step of the calculation. A further condition for stability is that the time intervals represented in the two steps of the simulation (equations 14 and 15) must be equal. The length of the time interval may differ from one pair of time steps to the next, but within a given pair, as used in equations 14 and 15, the two values of Δt must be kept the same. Finally, there must be an even number of total time steps; $\partial^2 h/\partial y^2$ must be simulated prior to $\partial^2 h/\partial x^2$ as often as $\partial^2 h/\partial x^2$ is simulated prior to $\partial^2 h/\partial y^2$.

If equations 14 and 15 are written out using the earlier notation we have

$$\frac{h_{i-1,j,n-1} + h_{i+1,j,n-1} - 2h_{i,j,n-1}}{(\Delta x)^2} + \frac{h_{i,j-1,n} + h_{i,j+1,n} - 2h_{i,j,n}}{(\Delta y)^2} = \frac{S}{T} \frac{h_{i,j,n} - h_{i,j,n-1}}{\Delta t} \quad (16)$$

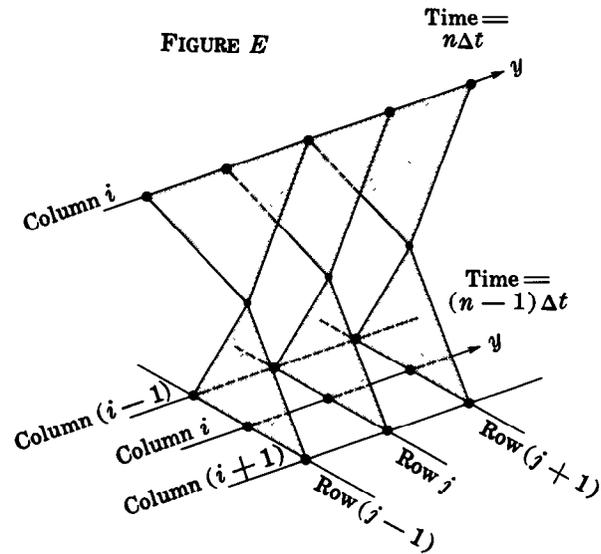
and

$$\frac{h_{i-1,j,n+1} + h_{i+1,j,n+1} - 2h_{i,j,n+1}}{(\Delta x)^2} + \frac{h_{i,j-1,n} + h_{i,j+1,n} - 2h_{i,j,n}}{(\Delta y)^2} = \frac{S}{T} \frac{h_{i,j,n+1} - h_{i,j,n}}{\Delta t} \quad (17)$$

Equation 16 involves three values of head along row j at time $(n-1)\Delta t$ and three values of head along column i at time $n\Delta t$. Let us assume that the head values for the earlier time, $(n-1)\Delta t$, have been calculated throughout the x, y plane and that we are concerned with calculation of head values for the time $n\Delta t$. Equation 16 then contains three known values of head, for the time $(n-1)\Delta t$ and three unknown, for the time $n\Delta t$. Since we have three unknowns in one equation, we will again need to use simultaneous equations. In this case the three unknowns occur along a single column; and by considering other equations which apply along this column we can develop a convenient method of solution.

Let us suppose that there are m nodes along column i and that the head is specified at the two end nodes by boundary conditions, but must be determined for all of the interior nodes. The first node is identified by the subscript $j=1$ (we assume that the x -axis, where $j=0$, lies outside the problem area); the final node is identified by the subscript $j=m$. Thus $h_{i,1,n}$ and $h_{i,m,n}$ are specified by boundary conditions, while $h_{i,2,n}$ through $h_{i,m-1,n}$ must be determined.

We can write an equation of the form of equation 16 for each interior node along column i . As we set up the equation at each node, we pick up three known values of head from the $(n-1)\Delta t$ "time plane"; these known values fall along a three-column band, as shown in figure E. Each equation also incorporates three values of head for the new time, $n\Delta t$, all lying along column i ; and when we have set up an equation of the form of equation 16 for each interior node along the column, we have a system of $m-2$ equations in $m-2$ unknowns, which can be solved simultaneously. The solution of this set of equations is undertaken independently from the solutions for adjacent columns in the mesh; thus, instead of dealing with a set of, say, 2,500 simultaneous equations in a 50 by 50 array, we deal in turn with separate sets of only 50 equations. Each of these sets corresponds to a column within the mesh; and



each is much easier to solve than the 2,500 equation set, not only because of the smaller number of equations, but also because a convenient order of computation is possible. We are able to utilize this order of computation through a technique developed by H. L. Thomas (1949) that is known as the Thomas algorithm.

To illustrate this method, we rearrange equation 16, putting the unknown values of head, corresponding to time $n\Delta t$, on one side, as follows:

$$\frac{h_{i,j-1,n}}{(\Delta y)^2} - \left(\frac{S}{T\Delta t} + \frac{2}{(\Delta y)^2} \right) h_{i,j,n} + \frac{h_{i,j+1,n}}{(\Delta y)^2} = - \frac{h_{i-1,j,n-1}}{(\Delta x)^2} - \left(\frac{S}{T\Delta t} - \frac{2}{(\Delta x)^2} \right) h_{i,j,n-1} - \frac{h_{i+1,j,n-1}}{(\Delta x)^2} \quad (18)$$

The right-hand side consists entirely of known terms, and it is convenient to replace this side of the equation by a single symbol, D_j , that is

$$D_j = - \frac{h_{i-1,j,n-1}}{(\Delta x)^2} - \left(\frac{S}{T\Delta t} - \frac{2}{(\Delta x)^2} \right) h_{i,j,n-1} - \frac{h_{i+1,j,n-1}}{(\Delta x)^2}, \quad (19)$$

The single subscript, j , is sufficient to designate D for our purposes. As suggested in figure E, the sequence of calculation is along the column i . At each node—that is, for each value of j —there is only one value of D , taken from the three-column band in the preceding time plane. We are limiting consideration here to one set of equations, corresponding to one column, and aimed at calculating the heads for one value of time; the subscripts designating the column and

time are therefore not required. Thus we can omit the subscripts i and n from the values of h on the left side of the equation. With these changes, equation 18 takes the form

$$A_j h_{j-1} + B_j h_j + C_j h_{j+1} = D_j \quad (20)$$

where, in the problem which we have set up

$$A_j = \frac{1}{(\Delta y)^2},$$

$$B_j = -\left(\frac{S}{T \Delta t} + \frac{2}{(\Delta y)^2} \right),$$

and

$$C_j = \frac{1}{(\Delta y)^2}.$$

The coefficients A , B , and C are constant for the problem which we have postulated. In some problems, however, where variation in T , S , or the node spacing is involved, they may vary from one node to another. To keep the discussion sufficiently general to cover such cases, the coefficients have been designated with the subscript j .

If we solve equation 20 for h_j , the central value of the three-node set represented in the equation, we obtain

$$h_j = \frac{D_j - A_j h_{j-1} - C_j h_{j+1}}{B_j} \quad (21)$$

h_1 the head at the initial node of the column, is specified by the boundary condition. We apply equation 21 to find an expression for h_2 ; this gives

$$h_2 = \frac{D_2 - A_2 h_1 - C_2 h_3}{B_2} \quad (22)$$

We rewrite this equation in the form

$$h_2 = g_2 - b_2 h_3 \quad (23)$$

where

$$g_2 = \frac{D_2 - A_2 h_1}{B_2} \quad (24)$$

and

$$b_2 = \frac{C_2}{B_2} \quad (25)$$

b_2 consists of known terms, and since h_1 is known, g_2 can be calculated; equation 23 thus gives us an equation for h_2 in terms of the next succeeding value of head, h_3 . If we can continue along the column, forming equations which give the head at each node in terms of that at the succeeding node—that is, which give h_j in terms of h_{j+1} —we will eventually reach the next to last node in the column, where we will have an equation for h_{m-1} in terms of h_m , the head at the last node. Then since h_m is known, from the boundary condition, we will be able to calculate h_{m-1} ; using this value of h_{m-1} we can calculate h_{m-2} , and so on back down the column, until finally we can calculate h_2 in terms of h_3 using equation 23. This is the basic idea of the Thomas algorithm. We now have to see whether we can in fact obtain expressions for each head, h_j , in terms of the succeeding head, h_{j+1} , along the column.

We first apply equation 21 to find an expression for h_3 obtaining

$$h_3 = \frac{D_3 - A_3 h_2 - C_3 h_4}{B_3} \quad (26)$$

To eliminate h_2 from this equation, we substitute from equation 23, obtaining

$$h_3 = \frac{D_3 - A_3 (g_2 - b_2 h_3) - C_3 h_4}{B_3} \quad (27)$$

Equation 27 is now solved for h_3 as follows

$$h_3 - \frac{A_3 b_2}{B_3} h_3 = \frac{D_3 - A_3 g_2 - C_3 h_4}{B_3}$$

$$h_3 = \frac{D_3 - A_3 g_2 - C_3 h_4}{B_3 \left(\frac{B_3 - A_3 b_2}{B_3} \right)}$$

or

$$h_3 = \frac{D_3 - A_3 g_2}{B_3 - A_3 b_2} - \frac{C_3}{B_3 - A_3 b_2} h_4 \quad (28)$$

Now again we have an equation of the form

$$h_3 = g_3 - b_3 h_4 \quad (29)$$

where here

$$g_3 = \frac{D_3 - A_3 g_2}{B_3 - A_3 b_2} \quad (30)$$

and

$$b_3 = \frac{C_3}{B_3 - A_3 b_2}. \quad (31)$$

Since g_2 and b_2 are known from the preceding step of the calculation (equation 24 and 25), g_3 and b_3 can be calculated, and equation 29 then gives us an expression for h_3 in terms of h_4 . In effect, we have eliminated h_2 from equation 26, so that h_3 is expressed in terms of the succeeding value of head alone.

If we continue this process, we find that at each step we can obtain an equation of the form

$$h_j = g_j - b_j h_{j+1} \quad (32)$$

relating the head at each node to that at the succeeding node; and we find that g_j and b_j can always be determined from the preceding values of g and b by equations of the form of equations 30 and 31. That is, we find that

$$g_j = \frac{D_j - A_j g_{j-1}}{B_j - A_j b_{j-1}} \quad (33)$$

and

$$b_j = \frac{C_j}{B_j - A_j b_{j-1}}. \quad (34)$$

These general formulas apply even to the calculation of g_2 and b_2 if we specify the starting conditions $g_1 = h_1$ and $b_1 = 0$.

In summary, then, we may start at node 1 and move up the column calculating values of g_j and b_j . At each node, these values are calculated by equations 33 and 34, using the preceding values, g_{j-1} and b_{j-1} , and using the coefficients A_j , B_j , and C_j and the term D_j .

Ultimately, at the next to last node of the column, g_{m-1} and b_{m-1} are calculated; then since h_m is known from the boundary condition, h_{m-1} can be calculated from equation 32. We then proceed back down the column, calculating the value of h_j at each node from

the value of h_{j+1} using equation 32, until finally a value for h_2 has been calculated and heads have been determined throughout the column.

The whole process is actually one of Gaussian elimination, taking advantage of a convenient order of calculation. The solution of the difference equation 16 is obtained directly for points along the column through this process; we are not dealing with an iterative technique which solves the set of algebraic equations by successive approximation. When the head has been calculated at all nodes along column i , the process is repeated for column $i+1$, and so on until the entire plane has been traversed.

In a sense, this process of calculation stands somewhere between the forward-difference technique and the backward-difference technique. In the forward-difference technique the head at every node, for a given time level, is computed independently from the heads at the four adjacent nodes for that time level; the technique of computation is said to be explicit. In the backward-difference technique, the calculation of the head at each node incorporates the heads at the four adjacent nodes for the same time level; the method of calculation is termed implicit. In the alternating-direction technique the calculation of the head at a given node, as we move along a column, incorporates the heads for that time level at the two adjacent nodes along the column, but not at the two adjacent nodes in the adjoining columns. The method of calculation, for this step, is said to be implicit along the columns, but explicit in the transverse direction, along the rows.

When the heads have been calculated everywhere throughout the plane by the process of traversing the columns, calculations for the following time, $(n+1)\Delta t$ are initiated using equation 17. The procedure followed is the same as that described above, except that the calculation now moves along rows, rather than along columns. This alternation of direction again, is necessary in order to insure the stability of the method of calculation.

Solution of the steady-state equation by iteration using the alternating-direction method of calculation

In their initial paper proposing the alternating-direction implicit procedure, Peaceman and Rachford point out that the technique of solving alternately along rows and columns can be used effectively to iterate the steady-state equation. That is, suppose we must deal with the problem considered in Section 16 and 25 of Part VII, and reviewed earlier in the present discussion, in which the steady-state equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (35)$$

is to be solved. In Section 25, we considered a technique of iteration, or relaxation, to solve this equation. In this technique we wrote the finite-difference approximation given in equation 6 as a simulation of equation 35; this gave

$$h_{i,j} = \frac{1}{4}(h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}). \quad (6)$$

To apply equation 6, we would move through the x, y plane replacing values of $h_{i,j}$ at each interior node by the average of the heads at the four surrounding nodes. At the end of one complete traverse of the plane we would have a set of values of $h_{i,j}$ which would be somewhat closer to satisfying equation 6 than were the values with which we started; and after several traverses, we would have a set of head values which would essentially satisfy equation 36 throughout the plane. This would be indicated by the fact that the values of $h_{i,j}$ obtained in each step would differ very little from those obtained in the preceding step.

Our objective here is to outline a more efficient technique of carrying out this itera-

tion process, based upon Peaceman and Rachford's method and the Thomas algorithm. We begin by introducing some nomenclature and notation. In our discussion of nonequilibrium problems, we spoke of "time planes"—that is, representations of the x, y plane in which the heads calculated for a given time were displayed. In discussing the solution of steady-state problems by iteration we can similarly speak of "iteration planes"—that is, representations of the x, y plane in which the values of head obtained after a certain number of iterations are displayed. Again, in our discussion of nonequilibrium problems we used the subscript n to designate the time level of a given head value— $h_{i,j,n}$ referred to a head value at the time $n\Delta t$. In a similar way, we will use a superscript m to denote the iteration level in the steady-state problem. $h_{i,j}^0$ will be used to designate the starting values of head, prior to any iterations; $h_{i,j}^1$ will indicate head values after one iteration—that is, the head values in the first iteration plane; and in general, $h_{i,j}^m$ will indicate head values after m iterations, or in the m th iteration plane.

Next we rewrite our approximation to equation 35 in a slightly different form. We rearrange equation 6 to give

$$h_{i-1,j} + h_{i+1,j} - 2h_{i,j} = -h_{i,j-1} - h_{i,j+1} + 2h_{i,j} \quad (36)$$

This can be obtained also by rewriting equation 35 in the form

$$\frac{\partial^2 h}{\partial x^2} = -\frac{\partial^2 h}{\partial y^2}$$

and then using the approximations given in equation 8 and 10 for $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$.

We are interested in applying equation 36 to calculate head values for a new iteration level, using head values from the preceding

iteration level. In the procedure which we will employ it is necessary to consider two successive iteration steps. Using the superscript notation described above, and using $\Delta_{xx}h$ and $\Delta_{yy}h$ to represent our approximations to $\partial^2 h/\partial x^2$ and $\partial^2 h/\partial y^2$ as in equations 8 and 10, the method of calculation may be summarized as follows

$$-\Delta_{yy}h^m = \Delta_{xx}h^{m-1} \quad (37)$$

and

$$\Delta_{xx}h^{m+1} = -\Delta_{yy}h^m \quad (38)$$

or, in the notation of equation 36,

$$\begin{aligned} -h_{i,j-1}^m - h_{i,j+1}^m + 2h_{i,j}^m \\ = h_{i-1,j}^{m-1} + h_{i+1,j}^{m-1} - 2h_{i,j}^{m-1} \end{aligned} \quad (39)$$

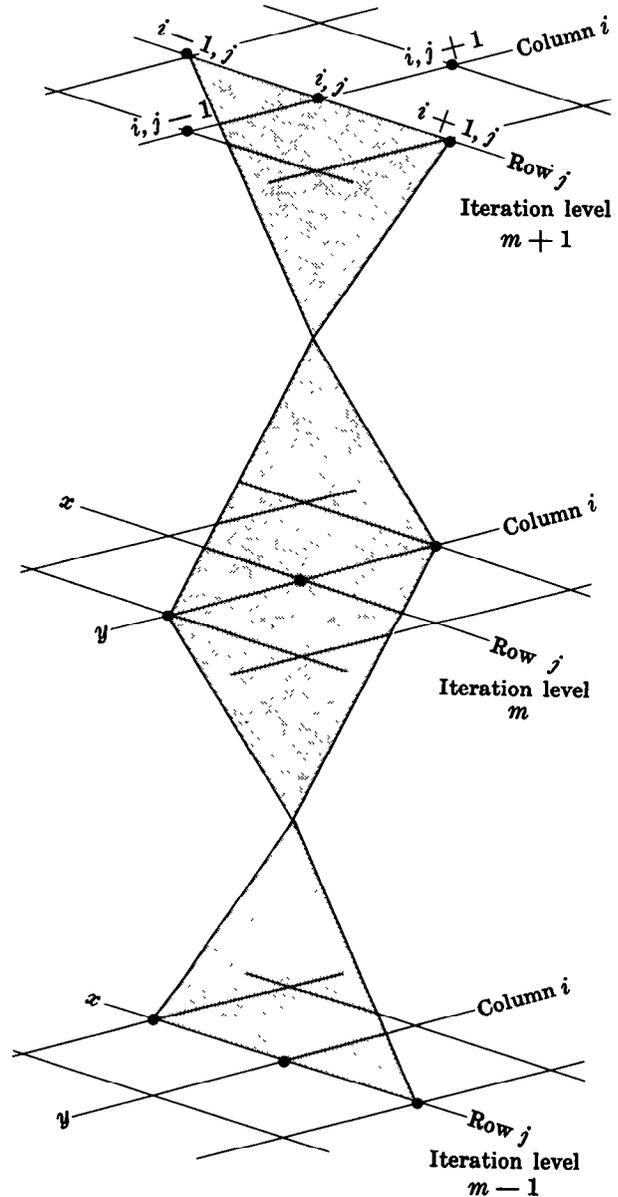
and

$$\begin{aligned} h_{i-1,j}^{m+1} + h_{i+1,j}^{m+1} - 2h_{i,j}^{m+1} \\ = -h_{i,j-1}^m - h_{i,j+1}^m + 2h_{i,j}^m \end{aligned} \quad (40)$$

As these equations indicate, the idea here is first to simulate $\partial^2 h/\partial x^2$ at one iteration level and $\partial^2 h/\partial y^2$ at the next; in the succeeding iteration, the order is reversed; $\partial^2 h/\partial y^2$ is simulated at the earlier iteration level, and $\partial^2 h/\partial x^2$ at the next. Figure D, which illustrated the simulation technique for the non-equilibrium problem, is reproduced as figure F, but with the time planes now relabeled as iteration planes. Equation 39 relates three values of head at iteration level m to three values at iteration level $m-1$; and, following the technique described above for the non-equilibrium case, we may move along column i in iteration plane m , at each node picking up three known values of h^{m-1} from a three column band in the preceding iteration plane, and thus generating a set of equations in which the unknowns are all values of h^m along column i .

As in the nonequilibrium case, the set of equations along a given column is solved directly by the Thomas algorithm—that is, by

FIGURE F



the process of Gaussian elimination outlined in equation 20 through 34. When this has been done for every column in the x, y plane, we have a new set of head values throughout the plane. These values, however do not necessarily constitute a solution to equation 35. The process we have described, of replacing the earlier head values with new values calculated through equation 39, accomplishes the same thing as the relaxation process of Section 25—it produces a new set of values which is closer to satisfying equation 35 than was the earlier set. This does not guarantee that the new set will constitute an acceptable solution. The test as to whether or not a solution has been found is carried out as in the relaxation technique of Section 25—

the values of head in iteration plane m are compared to those in iteration plane $m-1$. If the difference is everywhere negligible, equation 35 must be satisfied throughout the x, y plane; otherwise a new iteration must be initiated. In this new iteration we would utilize equation 40, moving along a row of the model to set up a system of equations for the head values along that row. As in the nonequilibrium problem this alteration of direction is necessary for stability. In summary then, we are utilizing an indirect iterative procedure of solution; but we use a direct method, Gaussian elimination, along each individual column or row, to move from one set of approximate head values to the next during the iterative process.

Backward-difference simulation: Solution by iteration using the alternating-direction method of calculation (iterative alternating-direction implicit procedure)

Peaceman and Rachford found that iteration of the steady-state equation by the alternating-direction procedure was considerably more efficient than the most rapid relaxation techniques that had been used prior to the time of their work. The use of the alternating-direction technique in this sense, as a method of iteration, has accordingly gained great popularity in recent years. As a method of solving the nonequilibrium equation 3, however, the alternating-direction implicit procedure, as embodied in equations 14 and 15 or 16 and 17, has not always proved advantageous. Although stability is assured, that is the calculation will not be affected by errors which necessarily increase in magnitude at each step, there is still a possibility for large error at any one time step and at any given node; and in many problems these errors have proved uncontrollable and unacceptable. This undesirable feature has inevitably led to renewed interest in the backward-difference formulation of equations 5 and 13. As we have noted,

solution by this method must generally be accomplished through iteration, for example using equation 7; the systems of simultaneous equations involved are usually too large to admit of an easy solution by direct methods. We have seen that the alternating-direction procedure of Peaceman and Rachford provides an effective method of iterating the steady-state equation; this suggests that the same technique may be used to iterate the backward-difference equation, 5 or 13. Equation 13, which utilized the abbreviated notation, is reproduced below

$$(\Delta_{xx}h)_n + (\Delta_{yy}h)_n = \frac{S}{T} \cdot \frac{h_{i,j,n} - h_{i,j,n-1}}{\Delta t} \quad (13)$$

$(\Delta_{xx}h)_n$ is an approximation to $\partial^2 h / \partial x^2$ at the time $n\Delta t$, while $(\Delta_{yy}h)_n$ is an approximation $\partial^2 h / \partial y^2$ at the time $n\Delta t$. We again introduce the superscript m to indicate the level of iteration; using this notation we rewrite equation 13 as it will be used in two

successive steps of the iteration process under consideration,

$$(\Delta_{xx}h)_n^{m-1} + (\Delta_{yy}h)_n^m = \frac{S}{T} \frac{h_{i,j,n}^m - h_{i,j,n-1}}{\Delta t} \quad (41)$$

$$(\Delta_{xx}h)_n^{m+1} + (\Delta_{yy}h)_n^m = \frac{S}{T} \frac{h_{i,j,n}^{m+1} - h_{i,j,n-1}}{\Delta t} \quad (42)$$

Several points about equations 41 and 42 should be noted carefully. The simulations of both $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$, in both equations, are made at time $n\Delta t$; and again, in both equations, $\partial h / \partial t$ is simulated by the change in head at node i, j from time $(n-1)\Delta t$ to time $n\Delta t$. In equation 41, $(\partial^2 h / \partial x^2)_{n\Delta t}$ is simulated at the $(m-1)$ th iteration level, whereas $(\partial^2 h / \partial y^2)_{n\Delta t}$ is simulated at the m th iteration level; $h_{i,j,n}$, in the simulation of the time derivative, is represented at the m th iteration level. In equation 42, $(\partial^2 h / \partial y^2)_{n\Delta t}$ is simulated at the m th iteration level, while $(\partial^2 h / \partial x^2)_{n\Delta t}$ is simulated at the $(m+1)$ th iteration level; $h_{i,j,n}$, in the simulation of the time derivative, is again represented at the higher iteration level,

which is here $m+1$. No iteration superscript is attached to $h_{i,j,n-1}$ the head at the preceding time level, in either equation. The iteration process is designed to compute heads for the new time level, $n\Delta t$, and in this process the head at the preceding time level is simply a constant; it retains the same value throughout the series of iterations.

Rewriting equation 41 using the expanded notation for $\Delta_{xx}h$ and $\Delta_{yy}h$ (as given in equations 8 and 10), we have

$$\begin{aligned} & \frac{h_{i-1,j,n}^{m-1} + h_{i+1,j,n}^{m-1} - 2h_{i,j,n}^{m-1}}{(\Delta x)^2} \\ & + \frac{h_{i,j-1,n}^m + h_{i,j+1,n}^m - 2h_{i,j,n}^m}{(\Delta y)^2} \\ & = \frac{S}{T} \frac{(h_{i,j,n}^m - h_{i,j,n-1})}{\Delta t} \end{aligned} \quad (43)$$

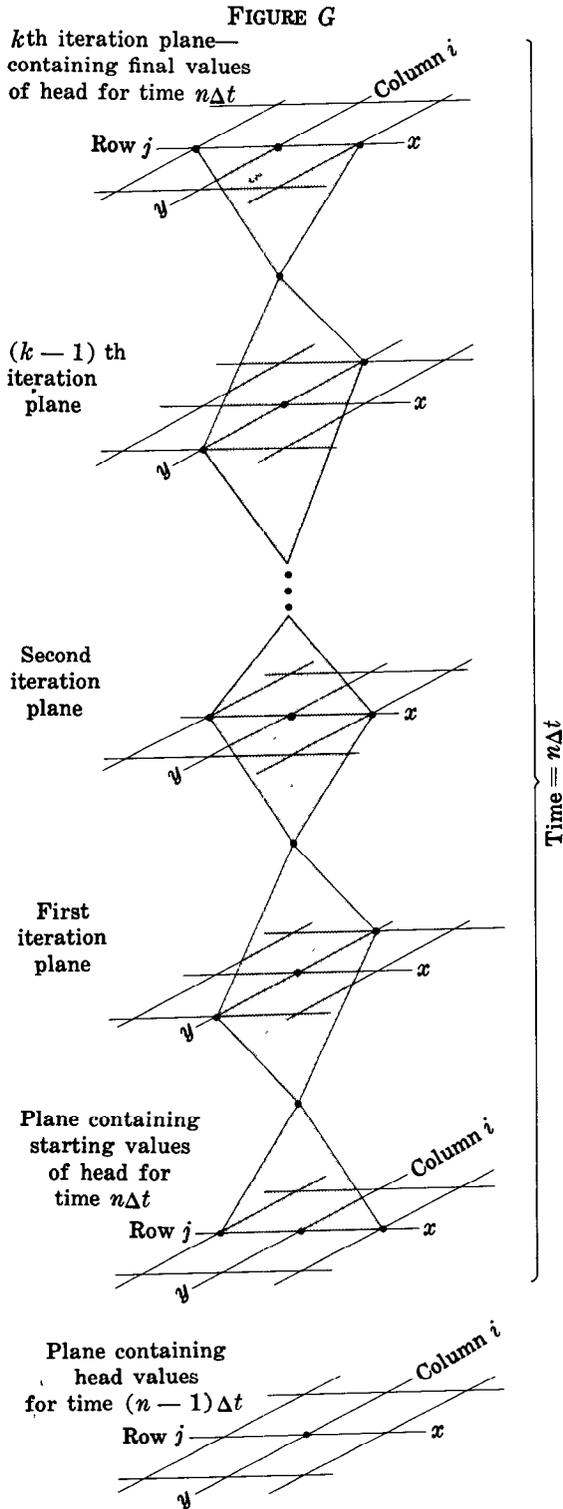
We wish to calculate head values at the new iteration level, m , on the basis of values which we already have for the preceding iteration level, $m-1$. We therefore rearrange equation 43, placing unknown terms on the left and known terms on the right. This gives

$$\begin{aligned} \frac{h_{i,j-1,n}^m}{(\Delta y)^2} + \frac{h_{i,j+1,n}^m}{(\Delta y)^2} - \left(\frac{S}{T\Delta t} + \frac{2}{(\Delta y)^2} \right) h_{i,j,n}^m = \\ - \frac{h_{i-1,j,n}^{m-1}}{(\Delta x)^2} - \frac{h_{i+1,j,n}^{m-1}}{(\Delta x)^2} + \frac{2}{(\Delta x)^2} h_{i,j,n}^{m-1} - \frac{S}{T\Delta t} h_{i,j,n-1} \end{aligned} \quad (44)$$

The unknown terms are the head values for iteration level m ; the known terms are the head values for the preceding iteration level, $m-1$, and one head value from the preceding time level, $n-1$. We may therefore proceed as in equation 19, replacing the entire right side by a single symbol, D_j , representing the known terms of the equation. We will then have an equation of the form of equation 20,

$$A_j h_{i-1,n}^m + B_j h_{i,n}^m + C_j h_{i+1,n}^m = D_j, \quad (45)$$

which can be solved by the Thomas algorithm, as outlined in equations 21-34. In the next step we utilize equation 42; here the unknown terms consist of three values of h for time $n\Delta t$ and iteration level $m+1$, while the known terms consist of three values of h for time $n\Delta t$ and iteration level m , and again one value of h for the time level $(n-1)\Delta t$. After this step, the heads which we obtain



are compared with those obtained in the preceding step. If the difference is everywhere negligible, the values of h^{m+1} are taken as a sufficiently close approximation to the heads for time $n\Delta t$.

It's important to note that while at each step we solve directly, (by Gaussian elimination, along columns or rows) to obtain a new set of head values, these new values do not generally constitute a solution to our differential equation. Rather, they form a new approximation to a solution, in a series of iterations which will ultimately produce an approximation close enough for our purposes. We may review the sequence of computation by referring to figure G, which illustrates the process of calculation schematically. The lowermost plane in the figure is a time plane, containing the final values of head for the preceding time level, $(n - 1)\Delta t$. The plane immediately above this contains the initial assumed values of head for the new time, $n\Delta t$; we use three values of head, h_{i-1,j,n^0} , h_{i,j,n^0} , and h_{i+1,j,n^0} from this plane, together with one value of head $h_{i,j,n-1}$ from the $n - 1$ time plane, on the right side of equation 44. On the left side of equation 44 we have three unknown values of head in the first iteration plane, $h_{i,j-1,n^1}$, h_{i,j,n^1} , and $h_{i,j+1,n^1}$. We set up equations of the form of equation 44 along the entire column i and solve by the Thomas algorithm (equations 21-34). We then repeat the procedure along all other columns, thus determining head values throughout the first iteration plane; these new head values constitute a somewhat closer approximation to the heads at time $n\Delta t$ than did the initial values. Next we set up a system of equations of the form of equation 42, arranged so that in each equation three head values from the first iteration plane and one from the $n - 1$ time plane form the known terms, while three head values from the second iteration plane from the unknown terms. If we rewrite equation 42 in the expanded notation and rearrange it so that the unknown terms appear on the left and the known terms on the right we have

$$\frac{h_{i-1,j,n}^{m+1}}{(\Delta x)^2} + \frac{h_{i+1,j,n}^{m+1}}{(\Delta x)^2} - \left(\frac{S}{T\Delta t} + \frac{2}{(\Delta x)^2} \right) h_{i,j,n}^{m+1} = -\frac{h_{i,j-1,n}^m}{(\Delta y)^2} - \frac{h_{i,j+1,n}^m}{(\Delta y)^2} + \frac{2}{(\Delta y)^2} h_{i,j,n}^m - \frac{S}{T\Delta t} h_{i,j,n-1} \quad (46)$$

Applying equation 46 between the first and second iteration planes, m would be taken as 1 and $(m+1)$ as 2. The four known terms on the right side of the equation would consist of three head values from the first iteration plane $h_{i,j-1,n}$, $h_{i,j,n}$, and $h_{i,j+1,n}$, and again one head value from the $n-1$ time plane, $h_{i,j,n-1}$. It is important to note that we return to the $n-1$ time plane—the lowermost plane in figure G—at each iteration level in the series, to pick up the constant values of $h_{i,j,n-1}$ that are used in simulating the time derivative. On the left side of equation 46 we would have the three unknown values of head corresponding to the new iteration level—that is, the second iteration plane). Again we would use the Thomas algorithm (equations 21–34) to solve for these new values of head throughout the plane. At the end of this solution procedure the head values in the second iteration plane are compared with those in the first iteration plane. If the difference is sufficiently small at all points, there is nothing to be gained by continuing to adjust the head values through further calculation—equation 3 is already approximately satisfied throughout the plane. If significant differences are noted, the procedure is continued until the differences between the head values obtained in successive iteration levels becomes negligible. At this point the heads for time $n\Delta t$ have been determined, and work is started on the next time step, computing heads for the time $(n+1)\Delta t$. Thus while direct solution and an alternating-direction feature both play a part in this procedure of calculation, the technique is basically one of iteration, in which, using the backward-difference formulation of equations 5 or 13, we progressively adjust head values for each time level until we arrive at a set of values which satisfies the equation. The method combines the advantages of the backward-

difference technique with the ease of computation of the alternating-direction procedure; it is the basis of many of the digital models presently used by the U.S. Geol. Survey. It is sometimes referred to as the iterative alternating-direction implicit procedure.

Prickett and Lonquist (1971) further modify this method of calculation by representing the central head value, $h_{i,j}$ only at the advanced iteration level; and by representing the head in the adjacent, previously processed column also at the advanced iteration level. That is, they do not simulate $\partial^2 h / \partial x^2$ and $\partial^2 h / \partial y^2$ in two distinct iteration planes, but rather set up the calculation as a relaxation technique, so that the new value of head at a given node is calculated on the basis of the most recently computed values of head in the surrounding nodes. They do, however, perform the calculations alternately along rows and columns using the Thomas algorithm.

In the discussions presented here we have treated transmissivity, storage coefficient, and the node spacings Δx and Δy , as constant terms in the x, y plane. In fact these terms can be varied through the mesh to account for heterogeneity or anisotropy in the aquifer or to provide a node spacing which is everywhere suited to the needs of the problem. Additional terms can be inserted into the equations to account for such things as pumpage from wells at specified nodes, retrieval of evapotranspiration loss, seepage into streams, and so on. Some programs have been developed which simulate three-dimensional flow (Freeze, 1971; Bredehoeft and Pinder, 1970; Prickett and Lonquist, 1971, p. 46); however, the operational problems encountered in three-dimensional digital modeling are sometimes troublesome.

The reader may now proceed to the programmed instruction of Part VIII.